UPR-1115-T, hep-th/0504136

Supersymmetric Kerr–Anti-de Sitter solutions

Mirjam Cvetič, Peng Gao and Joan Simón[1]

[1] *Department of Physics and Astronomy, University of Pennsylvania, Philadelphia, PA 19104, USA**

(Dated: February 1, 2008)

Abstract

We prove the existence of one quarter supersymmetric type IIB configurations that arise as non-trivial scaling solutions of the standard five dimensional Kerr-Anti-de Sitter black holes by the explicit construction of its Killing spinors. This neutral, spinning solution is asymptotic to the static anti-deSitter space-time with cosmological constant $-\frac{1}{\ell^2}$, it has two finite equal angular momenta $J_1 = \pm J_2$, mass $M = \frac{1}{\ell}\left(|J_1| + |J_2|\right)$ and a naked singularity. We also address the scaling limit associated with one half supersymmetric solution with only one angular momentum.

*Electronic address: cvetic@cvetic.hep.upenn.edu; gaopeng@sas.upenn.edu; jsimon@bokchoy.hep.upenn.edu

Contents

I. INTRODUCTION

In the recent year important progress has been made in constructing general R-charged, spinning solutions with non-zero cosmological constant in dimensions $D = 5$ [1, 2] and $D = 7$ [3]. These solutions are parameterized by a mass, three charges in $D = 5$ (two charges in $D = 7$) and all angular momenta set equal. While the most general solutions with all non-equal angular momenta are still elusive, these solutions [1, 2, 3] provide a useful framework to address their thermodynamics [4], supersymmetric (BPS) limits, such as those found in $D = 5$ [5, 6], and their global space-time structure. [For the earlier study of the BPS limits of charged spinning solutions in $D = 4$ see [7].] In particular, the BPS limits of these spinning charged solutions should play an important role in elucidating the field configurations in the dual conformal field theory. [For example, the singularity of the extremal limit of the single R-charge spacetime [8] is interpreted as a distribution of giant gravitons [9, 10, 11].]

On the other hand, the general neutral (vanishing R-charge sector) spinning solutions with cosmological constant in $D = 5$ dimensions were constructed in [12] and subsequently in all dimensions $D > 5$ in [13]. Their thermodynamics has been studied extensively in [14]. However, the study of their BPS limits, except in $D = 3$, seems to have led to negative conclusions [12]. On the other hand, there is an expectation that in the dual field theory there should be field configurations that in the strongly coupled limit represent BPS

configurations with the role of the spin being associated with the angular momentum. The purpose of this paper is to prove explicitly the existence of supersymmetric, asymptotically Anti-de Sitter (AdS) space-times with finite energy and finite angular momentum, but no R-charge.

The existence of such BPS configurations is not forbidden from a supersymmetry algebra point of view. In the analysis done by [7, 15, 16] in four dimensions, one can certainly saturate the Bogomolnyi type bound in the zero R-charge sector. In principle similar conclusions can be derived in five dimensions from the general analysis of the Bogomolnyi bounds presented in [4]. The question remains, though, as to whether one can find explicit classical configurations carrying these global charges, and to show explicitly that these, Bogomolnyi bound saturating configurations, are indeed supersymmetric.

While we believe the analysis of such BPS configurations can be generalized to non-equal angular momenta and other dimensions, we shall focus on the case in $D = 5$ and two angular momenta equal. In the last section we shall also discuss the case that corresponds to the supersymmetric configuration with only one angular momentum turned on.

Our strategy will be as follows. We shall work in five dimensions, or equivalently, in type IIB compactified on a 5-sphere. In the absence of R-charge, the simplest ansatz to consider is one in which only the five dimensional metric degrees of freedom are excited. In the full type IIB description, this is equivalent to work in a Freund–Rubin ansatz [17]. The general five-dimensional Kerr-AdS black holes [12] are then the appropriate configurations to consider, since they have the correct asymptotics and carry the correct charges. These are families of configurations characterised by three charges : the mass and two angular momenta, which are in one to one correspondence with respective three parameters $\{m,\, a,\, b\}$. In general these configurations are not supersymmetric.

In order to obtain supersymmetric configurations for this class of solutions, we shall analyse particular *scaling limits*, for which the charges (mass, two angular momenta) remain *finite*. The finite charges, associated with this scaled space-time saturate the BPS algebra bound. From a geometrical point of view, the scaling limit corresponds to pushing the horizon of the original Kerr-AdS black hole to infinity. As a result, one expects to find a naked singularity. Nevertheless in this rescaled space time the asymptotics corresponds to the AdS space time.

Our main result is the explicit construction of the Killing spinors for the space-time in

the scaling limit with two equal, finite angular momenta, thus proving explicitly that these are supersymmetric space-times corresponding to the the one quarter BPS configuration. The solution has a point-like naked singularity, and asymptotes to AdS. It corresponds to a Lorentzian Einstein-Sasaki manifold, which is in agreement with the general statement that any five dimensional Lorentzian space-time in a Freund–Rubin Ansatz is either locally AdS, or a Lorentzian Einstein-Sasaki manifold or a space-time, conformal to a pp-wave [18].

The paper is organized in the following way. In section II, we discuss the scaling limit in detail, write down the corresponding geometry, and present the Killing spinors that support our claim. The technical details are discussed in the appendix. In section III, we comment on the existence of a second scaling limit, giving rise to a potentially one-half BPS configuration with a single angular momentum turned on. We comment on the puzzles associated with this configuration if we keep the AdS asymptote and on the emergence of a spacetime being conformal to a pp-wave if the physical parameter scaling is done together with a rescaling of a "lightcone" coordinate.

II. SCALING LIMIT OF KERR–ANTI-DE SITTER BLACK HOLES

The starting point of our analysis is the general five-dimensional Kerr–Anti-de Sitter black hole [12] :

$$
\begin{aligned}
ds_5^2 = &-\frac{\Delta}{\rho^2}\left[dt - \frac{a\sin^2\theta}{\Xi_a}\,d\phi - \frac{b\cos^2\theta}{\Xi_b}\,d\psi\right]^2 + \frac{\Delta_\theta\,\sin^2\theta}{\rho^2}\left[a\,dt - \frac{r^2+a^2}{\Xi_a}\,d\phi\right]^2 \\
&+ \frac{\Delta_\theta\,\cos^2\theta}{\rho^2}\left[b\,dt - \frac{r^2+b^2}{\Xi_b}\,d\phi\right]^2 + \frac{\rho^2\,dr^2}{\Delta} + \frac{\rho^2\,d\theta^2}{\Delta_\theta} \\
&+ \frac{(1+r^2\ell^{-2})}{r^2\rho^2}\left[ab\,dt - \frac{b(r^2+a^2)\sin^2\theta}{\Xi_a}\,d\phi - \frac{a(r^2+b^2)\cos^2\theta}{\Xi_b}\,d\psi\right]^2 \ , \qquad \text{(II.1)}
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta &= \frac{1}{r^2}(r^2+a^2)(r^2+b^2)(1+r^2\ell^{-2}) - 2m \ , \\
\Delta_\theta &= 1 - a^2\ell^{-2}\cos^2\theta - b^2\ell^{-2}\sin^2\theta \ , \\
\rho^2 &= r^2 + a^2\cos^2\theta + b^2\sin^2\theta \ , \\
\Xi_a &= 1 - a^2\ell^{-2} \ , \quad \Xi_b = 1 - b^2\ell^{-2} \ .
\end{aligned} \qquad \text{(II.2)}
$$

Following the thermodynamical analysis done in [14], the conserved charges carried by this configuration (energy E and angular momenta $\{J_1,\ J_2\}$) are given by

$$E = \frac{\pi\, m\,(2\Xi_a + 2\Xi_b - \Xi_a\,\Xi_b)}{4\Xi_a^2\,\Xi_b^2}\ ,\quad J_1 = \frac{\pi\, m\, a}{2\Xi_a^2\,\Xi_b}\ ,\quad J_2 = \frac{\pi\, m\, b}{2\Xi_a\,\Xi_b^2}\ . \tag{II.3}$$

One of the motivations for this work is the observation that in the above expressions there is a non-trivial *scaling limit* of the parameters $\{m,\ a,\ b\}$, for which this configuration that keeps all charges (II.3) *finite* and it saturate the Bogomolnyi bound. This scaling limit corresponds to:

$$a\ ,b \to \ell\ ,\quad M \equiv \frac{m}{\Xi^3}\ ,\text{fixed}\ . \tag{II.4}$$

Specifically, we take the limit, for which a and b approach ℓ at the same rate. In this case $\Xi \equiv \Xi_a = \Xi_b \to 0$. The novelty in the above scaling limit is that we allow ourselves to scale the mass parameter $m \to 0$ as both angular momentum parameters a and b reach their extremal values ℓ. After this scaling limit, the physical charges of the configuration are *finite* and satisfy the relation :

$$E = \pi\, M\ ,\quad J_1 = J_2 = \frac{\ell}{2}E\ ,$$

and thus

$$E \cdot \ell = J_1 + J_2\ . \tag{II.5}$$

Employing the supersymmetry algebra (with R-charges turned off), as done in [4], the the eigenvalues $\{\lambda\}$ of the Bogomolnyi matrix are given by:

$$\lambda = E \pm \frac{J_1}{\ell} \pm \frac{J_2}{\ell}\ ,$$

where all signs are uncorrelated. It is now obvious that (II.5) indeed saturates a supersymmetry bound, and that the number of preserved supercharges, should they exist, would be one quarter of the original ones, since there is a single vanishing eigenvalue when the limit (II.4) is considered. In the following subsection, we will explicitly prove that our configuration (II.7) is a one quarter BPS one by constructing its Killing spinors, thus matching the purely algebraic analysis mentioned above.

Concerning the nature of the metric after the scaling (II.4), it can most easily be captured

by working in a coordinate system which is non-rotating at infinity [12]. This is defined by

$$\begin{aligned}
\Xi_a\, y^2\, \sin^2\hat{\theta} &= (r^2+a^2)\, \sin^2\theta\ ,\\
\Xi_b\, y^2\, \cos^2\hat{\theta} &= (r^2+b^2)\, \cos^2\theta\ ,\\
\hat{\phi} &= \phi + a\ell^{-2}\, t\ ,\\
\hat{\psi} &= \psi + b\ell^{-2} t\ .
\end{aligned} \tag{II.6}$$

Working out the change of variables, and taking the scaling limit (II.4), the final metric can be written as

$$\begin{aligned}
ds_5^2 = -\left(\frac{y^2}{\ell^2}+1\right) dt^2 + \frac{2M}{y^2}\left(dt - \ell\sin^2\hat{\theta} d\hat{\Phi} - \ell\cos^2\hat{\theta} d\hat{\Psi}\right)^2 \\
+ \frac{dy^2}{1+\frac{2M\ell^2}{y^4}+\frac{y^2}{\ell^2}} + y^2\left(d\hat{\theta}^2 + \sin^2\hat{\theta} d\hat{\Phi}^2 + \cos^2\hat{\theta} d\hat{\Psi}^2\right)\ .
\end{aligned} \tag{II.7}$$

This metric has a curvature singularity at $y=0$. It is naked due to the absence of horizons, and has no closed time-like curves. Let us notice that from the original Kerr-AdS black hole perspective, the scaling limit (II.4) is effectively pushing the horizon of the black hole to infinity, since the mass parameter m, responsible for its finiteness of the horizon, was scaled to zero. From this perspective, it is natural to expect that the space-time has a naked singularity. Note however, that the scaling still allows us to asymptotically ($y\to\infty$) reach the AdS space-time, in static coordinates.

[It turns out that the above metric can be obtained as a lorentzianisation, i.e. $t\to it$ and $\ell\to -i\ell$, of specific Einstein-Sasaki metrics obtained in [19, 20]. Discussions along this direction can be found in [21] [23]. Our motivation, though, was primarily driven from a purely Lorentzian point of view.]

A. Supersymmetry

Since the configuration (II.7) does not carry any R-charge, the existence of supersymmetry can be answered by analysing the existence of non-trivial solutions to the Killing spinor equation. The latter is the standard Killing spinor equation for spaces with a negative cosmological constant :

$$\hat{\nabla}_\mu\eta = \left(\nabla_\mu + \frac{1}{2\ell}\gamma_\mu\right)\eta = 0\ , \tag{II.8}$$

where $\nabla_\mu = \partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}$ stands for the standard covariant derivative on spinors.

It will turn out very convenient for our purposes to analyse the set of constraints imposed by the first two integrability conditions associated with (II.8) :

$$
\begin{aligned}
[\hat{\nabla}_\mu,\, \hat{\nabla}_\nu]\eta = &\ \ 0\ , \\
\big[\hat{\nabla}_\lambda,\, [\hat{\nabla}_\mu,\, \hat{\nabla}_\nu]\big]\eta = &\ \ 0\ .
\end{aligned}
\tag{II.9}
$$

These are equivalent, respectively, to [22]

$$
\begin{aligned}
C_{\mu\nu\rho\sigma}\gamma^{\rho\sigma}\eta = &\ \ 0\ , \\
(\nabla_\lambda C_{\mu\nu\rho\sigma})\,\gamma^{\rho\sigma}\eta + \frac{1}{\ell}C_{\mu\nu\lambda\rho}\gamma^\rho\eta = &\ \ 0\ ,
\end{aligned}
\tag{II.10}
$$

where $C_{\mu\nu\rho\sigma}$ stands for the components of the Weyl tensor.

Out of these equations, we derive two inequivalent algebraic projections conditions :

$$
(f_1(y)\gamma_{12} - f_2(y)\gamma_{35} - f_3(y)\gamma_{24})\,\eta \ = \ 0\ , \tag{II.11}
$$

$$
i\gamma_{23}\eta \ = \ -\eta\ , \tag{II.12}
$$

where the three scalar functions $f_i(y)$ $i = 1, 2, 3$ are defined by

$$
f_1(y) = \frac{\sqrt{y^6 + \ell^2 y^4 + 2M\ell^4}}{y\,\sqrt{y^4 + \ell^2 y^2 - 2M\ell^2}}\ , \quad f_2(y) = \frac{\ell}{y}\ , \quad f_3(y) = \frac{y^2 + \ell^2}{\sqrt{y^4 + \ell^2 y^2 - 2M\ell^2}}\ ,
$$

and we already used the fact that the five dimensional gamma matrix γ_5 can be expressed as $i\gamma_{1234}$.

That these equations are indeed projection conditions can be trivially realised for (II.12), whereas for (II.11), it follows from the identity

$$
(f_1(y))^2 + (f_2(y))^2 = (f_3(y))^2\ .
$$

Notice that the subspace of solutions of each projection condition are orthogonal, which implies that only one quarter of all available spinors do satisfy both equations. This orthogonality can be exposed in a more manifest way by realising that the space of solutions to (II.11) is equivalent to the space of solutions of

$$
(f_3(y)\gamma_{14} + if_2(y)\gamma_4)\,\eta = f_1(y)\eta\ .
$$

It is now clear that both γ_{14} and $i\gamma_4$ commute with $i\gamma_{23}$.

In the appendix A, present the explicit resolution of the Killing spinor equations (II.8). It is proved there that the answer is given by

$$
\Psi = e^{-it/2\ell}\tilde{\Psi}(y)\ , \tag{II.13}
$$

where the spinor $\tilde{\Psi}(y)$ satisfies the differential equation

$$\frac{d\tilde{\Psi}}{dy} + \frac{1}{2}\frac{2M\ell^2(\ell^2+3y^2) - y^4(\ell^2+y^2)}{(y^4+y^2\ell^2-2M\ell^2)\sqrt{y^6+y^4\ell^2+2M\ell^4}}\gamma_{14}\,\tilde{\Psi} = 0 \,, \tag{II.14}$$

and fulfills both integrability conditions (II.11) and (II.12). Notice that this differential equation is regular at $y = 0$, where the naked singularity is.

III. DISCUSSION

In this note, we have analysed the scaling limit (II.4) of the general five dimensional Kerr-AdS black hole, and we obtained a singular, one quarter supersymmetric space-time configuration with finite energy and two equal, finite angular momenta. The singularity is point-like; it would be interesting to understand whether there is any source in string theory that could provide a physical interpretation of this singularity, analogous to the R-charged, non-spinning BPS configuration for the superstar [8]. Such a microscopical understanding of the solutions, studied in this paper, would in turn clarify whether this singular space-time is actually a solution of the full string theory.

Besides the scaling limit analysed previously, inspection of the thermodynamical quantities (II.3) suggests the possibility of a second, inequivalent scaling limit given by

$$a \to \ell \,, \quad M \equiv \frac{m}{\Xi_a^2} \,, b \text{ fixed} \,. \tag{III.1}$$

The conserved charges for such a configuration are finite and satisfy the following relations :

$$E = \frac{\pi}{2}\frac{M}{\Xi_b} \,, \quad J_a = \ell\, E \,, \quad J_b = 0 \,.$$

Notice that the second angular momentum was sent to zero even though the parameter b was kept fixed and different from ℓ. Thus, this second scaling limit (III.1) formally satisfies the identity

$$E \cdot \ell = J_a \,. \tag{III.2}$$

Employing the supersymmetry algebra of five dimensional gauged supergravity, one derives that (III.2) saturates the Bogomolnyi bound, corresponding to two equal Bogomolnyi matrix eigenvalues (II). Therefore one expects that in the scaling limit there is a configuration that preserves one half of the supersymmetry. In order to ensure that such a supersymmetric

configuration is indeed obtained from a scaling limit of the metric (II.1), one follows the arguments given in [4]. Namely, the existence of such one half supersymmetric configuration would imply the existence of two Killing vectors, constructed out of the corresponding Killing spinors. For that purpose, one employs the four Killing vectors of the five-dimensional AdS space-time:

$$K_{\pm\pm} = \partial_T + \ell^{-1}\left(\eta_1 \partial_{\hat{\phi}} + \eta_2 \partial_{\hat{\psi}}\right) \ , \tag{III.3}$$

where both $\{\eta_1,\, \eta_2\}$ are uncorrelated signs. [For the five-dimensional AdS space-time these four Killing vectors have a negative norm everywhere, and thus give rise to maximal supersymmetry.] We employ these four independent Killing vectors to analyse the norm of these Killing vectors for the space-time (II.1) in the above scaling limit.

Using the change of coordinates (II.6), we can rewrite these vectors in terms of the asymptotically rotating frame coordinate system, and compute their norms in the original metric (II.1) while taking the corresponding scaling limit (III.1). The four Killing vectors are :

$$K_{\eta_1\eta_2} = \partial_t + \eta_1 \ell^{-1}\left(1 - \eta_1\, a\ell^{-1}\right)\partial_\phi + \eta_2 \ell^{-1}\left(1 - \eta_2\, b\ell^{-1}\right)\partial_\psi \ , \tag{III.4}$$

and their norms are given by :

$$g(K_{+\eta_2},\, K_{+\eta_2}) \;=\; -1 \ , \tag{III.5}$$

$$g(K_{-\eta_2},\, K_{-\eta_2}) \;=\; -1 + \frac{8M\, \sin^4\theta}{r^2 + \ell^2\, \cos^2\theta + b^2 \sin^2\theta} \ . \tag{III.6}$$

Thus, we observe that the pair of Killing vectors $K_{+\eta_2}$ are everywhere causal, whereas the causality of the other pair $K_{-\eta_2}$ depends on the value of b. It is clear that for $b = 0$, the causal character of the corresponding vectors will be flipped somewhere in space-time, whereas for $b \neq 0$, such property depends on the physical ratio M/b^2. Therefore, this analysis substantiates the potential existence of a one-half supersymmetric solution in the above scaling limit.

Despite these algebraic facts, if one attempts to construct a finite metric (even for the $b = 0$ case) with the right AdS asymptotics, and the right cross-terms to describe non-trivial angular momentum, one apparently seems to require to scale M to infinity, in which case the limiting spacetime would have infinite charges.

A different possibility can arise if one changes the asymptotics of the corresponding scaled spacetime. Let us consider the $b = 0$ case in (III.1) and introduce the following "light-cone"

coordinates:

$$x^+ = t - a\hat{\phi}\,, \qquad x^- = t + a\hat{\phi}\,, \tag{III.7}$$

where $\hat{\phi} = \phi + a\,\ell^{-1}\,t$ is the angular coordinate in the asymptotically static space-time. If the scaling limit (III.1), namely:

$$a = \ell - \ell\,\epsilon\,, \quad m = M(1 - a^2\ell^{-2})^2 \to 4\,M\,\epsilon^2\,, \quad \epsilon \to 0\,, \tag{III.8}$$

is done together with the rescaling of the x^- light-cone coordinate

$$\hat{x}^- = \frac{x^-}{2\epsilon}\,, \quad \epsilon \to 0, \tag{III.9}$$

one obtains, as a consequence, that the Kerr-AdS metric (II.1) becomes:

$$\begin{aligned} ds^2 \;=\; & \frac{2M\ell^{-2}\sin^4\theta + \frac{1}{4}(\cos^2\theta + r^2\ell^{-2})[-\cos^2\theta\,(1+2r^2\ell^{-2}) + r^2\ell^{-2}]}{\cos^2\theta + r^2\ell^{-2}}\,dx^+\,dx^+ \qquad \text{(III.10)} \\ & -\;\sin^2\theta(1+r^2\ell^{-2})\,dx^+\,d\hat{x}^- \;+\; \frac{\cos^2\theta + r^2\ell^{-2}}{(1+r^2\ell^{-2})^2}\,dr^2 \;+\; \frac{\cos^2\theta + r^2\ell^{-2}}{\sin^2\theta\,\ell^{-2}}\,d\theta^2 \;+\; r^2\cos^2\theta\,d\psi^2\,. \end{aligned}$$

If the above metric is supersymmetric, by the general results in [18], it should be conformal to a five dimensional pp-wave. Thus, we expect the existence of a coordinate transformation making this fact manifest.

We would want to conclude that for Kerr AdS black holes in other dimensions [13] analogous scaling limits to the ones considered here, with all angular momenta turned on would have a straightforward generalization. We expect solutions with finite energy and all angular momenta equal to be singular, but asymptotic to AdS, and to preserve some amount of supersymmetry.

Acknowledgments

We would like to thank Ofer Aharony, Joe Minahan, Hong Lü and Chris Pope for useful discussions and comments. The work is supported by the Department of Energy under Grant No. DE-FG02-95ER40893, by the National Science Foundation under Grant Nos. INT02-03585 (MC), PHY-0331728 (JS) and OISE-0443607 (JS) as well as by the Fay R. and Eugene L. Langberg Chair (MC).

APPENDIX A: ANALYSIS OF KILLING SPINOR EQUATIONS

In this appendix, we shall provide the explicit details giving rise to the Killing spinors associated to the scaling solution (II.7). Thus, we need first to compute the spin connection associated with this metric. Working with the vielbein basis :

$$\begin{aligned}
e^1 &= h_2\, dt + 2M\frac{\ell}{y^2} h_2^{-1}\left(\sin^2\theta\, d\phi + \cos^2\theta\, d\psi\right) \ , \\
e^2 &= \frac{y}{2}\sin(\phi+\psi)\, \sin 2\theta\, \left(d\phi - d\phi\right) + y\cos(\phi+\psi)\, d\theta \ , \\
e^3 &= \frac{y}{2}\cos(\phi+\psi)\, \sin 2\theta\, \left(d\psi - d\phi\right) + -y\sin(\phi+\psi)\, d\theta \ , \\
e^4 &= h_3\left(\sin^2\theta\, d\phi + \cos^2\theta\, d\psi\right) \ , \\
e^5 &= h_1\, dy \ ,
\end{aligned} \tag{A.1}$$

where we introduced the set of functions

$$h_2(y) = \left(1 - 2\frac{M}{y^2} + \frac{y^2}{\ell^2}\right)^{1/2} \ , \ h_3(y) = \left(\frac{y^6 + 2M\ell^4 + \ell^2 y^4}{y^4 + \ell^2 y^2 - 2M\ell^2}\right)^{1/2} \ ,$$

$$h_1(y)\cdot h_2(y)\cdot h_3(y) = y \ ,$$

the spin connection $\omega^a{}_b$ solving the algebraic equation

$$de^a + \omega^a{}_b \wedge e^b = 0 \ ,$$

is given by

$$\begin{aligned}
\omega^1{}_2 &= -\frac{2M\ell}{y^4\, h_2}\, e^3 \ , \ \omega^1{}_3 = \frac{2M\ell}{y^4\, h_2}\, e^2 \ , \\
\omega^1{}_4 &= \frac{2M\ell}{y^4}\left[1 + y\frac{h_2'}{h_2}\right] e^5 = F(y)\, e^5 \\
\omega^1{}_5 &= \frac{h_2'}{h_2\, h_1}\, e^1 - F(y)\, e^4 \ , \\
\omega^2{}_3 &= -\frac{2M\ell}{y^4\, h_2}\, e^1 + \left(\frac{h_3}{y^2} - \frac{2}{h_3}\right) e^4 \ , \\
\omega^2{}_4 &= \frac{h_3}{y^2}\, e^3 \ , \ \omega^3{}_4 = -\frac{h_3}{y^2}\, e^2 \ , \\
\omega^2{}_5 &= \frac{1}{y\, h_1}\, e^2 \ , \ \omega^3{}_5 = \frac{1}{y\, h_1}\, e^3 \ , \\
\omega^4{}_5 &= \frac{h_3'}{h_3\, h_1}\, e^4 + F(y)\, e^1 \ ,
\end{aligned} \tag{A.2}$$

where the function $F(y)$ appearing in the equations was defined in the second line above and all primes indicate derivative with respect to y.

We are now at a position to study the different components of the Killing spinor equation

$$\nabla_\mu \Psi = -\frac{1}{2\ell}\gamma_\mu \Psi \ .$$

First, consider the time component ($\mu = t$) equation :

$$\partial_t \Psi + \frac{1}{2}\left(h_2' h_1^{-1}\,\gamma_{15} + F(y) h_2\,\gamma_{45} - 2M\frac{\ell}{y^4}\,\gamma_{23}\right)\,\Psi = -\frac{1}{2\ell} h_2\,\gamma_1\,\Psi \ . \tag{A.3}$$

This is an equation that can be immediately integrated in terms of an exponential function, due to the Killing vector nature of ∂_t. Instead of proceeding in that way, we shall study the matrix acting on the Killing spinor whenever the two integrability conditions (II.11) and (II.12) are satisfied. It can be shown that

$$\left(h_2' h_1^{-1}\,\gamma_{15} + F(y) h_2\,\gamma_{45} - 2M\frac{\ell}{y^4}\,\gamma_{23} + \frac{1}{\ell} h_2\,\gamma_1\right)\,\Psi = \frac{i}{\ell}\,\Psi \ .$$

Thus, whenever the integrability conditions are satisfied, we can integrate (A.3) :

$$\Psi = e^{-it/2\ell}\tilde{\Psi} \ . \tag{A.4}$$

Let us consider next the $\mu = \theta$ component :

$$\partial_\theta \Psi + \left(M\frac{\ell}{y^3\,h_2}\,\gamma_{13} - \frac{h_3}{2y}\,\gamma_{34} + \frac{2}{h_1}\,\gamma_{25} + \frac{y}{2\ell}\,\gamma_2\right)\,e^{-(\phi+\psi)\gamma_{23}}\,\Psi = 0 \ . \tag{A.5}$$

It is amusing to realise that whenever both integrability conditions (II.11) and (II.12) are satisfied, the following identity holds :

$$\left(M\frac{\ell}{y^3\,h_2}\,\gamma_{13} - \frac{h_3}{2y}\,\gamma_{34} + \frac{2}{h_1}\,\gamma_{25} + \frac{y}{2\ell}\,\gamma_2\right)\,\Psi = 0 \ . \tag{A.6}$$

Therefore, we conclude the Killing spinor Ψ is independent of the θ angular variable ($\partial_\theta \Psi = 0$).

Let us jointly consider the two components involving the two angular variables $\{\phi,\,\psi\}$. For the first one, we have

$$\begin{aligned}\partial_\phi \Psi &+ \frac{1}{2}\sin 2\theta\left[\frac{M\,\ell}{y^3\,h_2}\,\gamma_{12} - \frac{h_3}{2y}\,\gamma_{24} - \frac{1}{2h_1}\,\gamma_{35} - \frac{y}{2\ell}\,\gamma_3\right]\,e^{-(\phi+\psi)\gamma_{23}}\,\Psi \\ &+ \frac{1}{2}\sin^2\theta\left[\left(\frac{2M\,\ell}{y^3}\,\frac{h_3}{h_2}\,h_2' - h_3\,F(y)\right)\gamma_{15} + \left(\left(\frac{h_3}{y}\right)^2 - 2 - \left(\frac{2M\,\ell}{y^3\,h_2}\right)^2\right)\gamma_{23}\right. \\ &\left. + \left(\frac{2M\,\ell}{y^2\,h_2}\,F(y) + \frac{h_3'}{h_1}\right)\gamma_{45} + \frac{2M}{y^2\,h_2}\,\gamma_1 + \frac{h_3}{\ell}\,\gamma_4\right]\,\Psi\end{aligned} \tag{A.7}$$

whereas for the second one:

$$\partial_\psi \Psi - \frac{1}{2}\sin 2\theta \left[\frac{M\,\ell}{y^3\,h_2}\,\gamma_{12} - \frac{h_3}{2y}\,\gamma_{24} - \frac{1}{2h_1}\,\gamma_{35} - \frac{y}{2\ell}\,\gamma_3 \right] e^{-(\phi+\psi)\gamma_{23}}\,\Psi$$
$$+ \frac{1}{2}\cos^2\theta \left[\left(\frac{2M\,\ell}{y^3}\,\frac{h_3}{h_2}\,h_2' - h_3\,F(y) \right)\gamma_{15} + \left(\left(\frac{h_3}{y}\right)^2 - 2 - \left(\frac{2M\,\ell}{y^3\,h_2}\right)^2 \right)\gamma_{23} \right.$$
$$\left. + \left(\frac{2M\,\ell}{y^2\,h_2}\,F(y) + \frac{h_3'}{h_1} \right)\gamma_{45} + \frac{2M}{y^2\,h_2}\,\gamma_1 + \frac{h_3}{\ell}\,\gamma_4 \right] \Psi\ . \quad \text{(A.8)}$$

It is important to realise that due to the identity (A.6), the matrix multiplying the $\sin 2\theta$ terms in both equations (A.7) and (A.8) vanishes identically. We are thus only left to evaluate the matrix multiplying both $\sin^2\theta$ and $\cos^2\theta$. Such matrix can be written, after using the second integrability condition (II.12) as

$$A(y)\gamma_4 + i\left(\left(\frac{h_3}{y}\right)^2 - 2 - \left(\frac{2M\,\ell}{y^3\,h_2}\right)^2 \right) + B(y)\gamma_1\ , \quad \text{(A.9)}$$

where the following definitions and identities hold :

$$\begin{aligned} A(y) &= \frac{2M\ell}{y^3}\frac{h_2'}{h_2\,h_3} - h_3\,F(y) + \frac{h_3}{\ell} = \frac{y^4 - 2M\ell^2}{\ell\,y^3}\,f_1(y)\ , \\ B(y) &= \frac{2M\ell}{y^2\,h_2}\left(F(y) + \frac{1}{\ell}\right) + \frac{h_3'}{h_1} = \frac{y^4 - 2M\ell^2}{\ell\,y^3}\,f_3(y)\ . \end{aligned}$$

Since the second integrability condition (II.11) is equivalent to

$$\left(f_1(y)\gamma_4 + f_3(y)\gamma_1\right)\Psi = i\frac{\ell}{y}\Psi\ ,$$

the matrix $A(y)\gamma_4 + B(y)\gamma_1$, when acting on the Killing spinor Ψ satisfies the identity

$$\left(A(y)\gamma_4 + B(y)\gamma_1\right)\Psi = i\frac{y^4 - 2M\ell^2}{y^4}\Psi\ .$$

It turns out that the above term is minus the second term in (A.9). Therefore, the sum of all matrices acting on Killing spinors appearing in (A.7) and (A.8) vanish whenever both integrability conditions (II.11) and (II.12) are satisfied. We conclude the Killing spinor is also independent of the angular variables $\{\phi,\,\psi\}$.

Finally, let us focus on the radial ($\mu = y$) component equation :

$$\partial_y\Psi + \frac{h_1}{2}\left(F(y)\,\gamma_{14} + \frac{1}{\ell}\,\gamma_5\right)\Psi = 0\ . \quad \text{(A.10)}$$

When using the partial integration (A.4) and the second integrability condition (II.12), we learn that

$$\frac{d\tilde{\Psi}}{dy} + \frac{h_1}{2}\left(F(y) - \frac{1}{\ell}\right)\gamma_{14}\tilde{\Psi} = 0\ .$$

The latter is a first order differential equation that can always be integrated. Evaluating both functions $F(y)$ and $h_1(y)$, its explicit expression is :

$$\frac{d\tilde{\Psi}}{dy} + \frac{1}{2}\frac{2M\ell^2(\ell^2+3y^2) - y^4(\ell^2+y^2)}{(y^4+y^2\ell^2-2M\ell^2)\sqrt{y^6+y^4\ell^2+2M\ell^4}}\,\gamma_{14}\,\tilde{\Psi} = 0\ , \tag{A.11}$$

To sum up, we have proved, by explicit construction, the existence of non-trivial Killing spinors for the background (II.7). For the reasons discussed in the main text, we conclude that such a background preserves one quarter of the supersymmetry.

---

[1] M. Cvetič, H. Lü, and C. N. Pope, "Charged kerr-de sitter black holes in five dimensions," *Phys. Lett.* **B598** (2004) 273–278, hep-th/0406196.

[2] M. Cvetič, H. Lü, and C. N. Pope, "Charged rotating black holes in five dimensional u(1)**3 gauged n = 2 supergravity," *Phys. Rev.* **D70** (2004) 081502, hep-th/0407058.

[3] Z. W. Chong, M. Cvetič, H. Lü, and C. N. Pope, "Charged rotating black holes in four-dimensional gauged and ungauged supergravities," hep-th/0411045.

[4] M. Cvetič, G. W. Gibbons, H. Lü, and C. N. Pope, "Rotating black holes in gauged supergravities: Thermodynamics, supersymmetric limits, topological solitons and time machines," hep-th/0504080.

[5] D. Klemm and W. A. Sabra, "General (anti-)de sitter black holes in five dimensions," *JHEP* **02** (2001) 031, hep-th/0011016.

[6] J. B. Gutowski and H. S. Reall, "General supersymmetric ads(5) black holes," *JHEP* **04** (2004) 048, hep-th/0401129.

[7] V. A. Kostelecky and M. J. Perry, "Solitonic black holes in gauged n=2 supergravity," *Phys. Lett.* **B371** (1996) 191–198, hep-th/9512222.

[8] R. C. Myers and O. Tafjord, "Superstars and giant gravitons," *JHEP* **11** (2001) 009, hep-th/0109127.

[9] J. McGreevy, L. Susskind, and N. Toumbas, "Invasion of the giant gravitons from anti-de sitter space," *JHEP* **06** (2000) 008, hep-th/0003075.

[10] M. T. Grisaru, R. C. Myers, and O. Tafjord, "Susy and goliath," *JHEP* **08** (2000) 040, hep-th/0008015.

[11] A. Hashimoto, S. Hirano, and N. Itzhaki, "Large branes in ads and their field theory dual," *JHEP* **08** (2000) 051, `hep-th/0008016`.

[12] S. W. Hawking, C. J. Hunter, and M. M. Taylor-Robinson, "Rotation and the ads/cft correspondence," *Phys. Rev.* **D59** (1999) 064005, `hep-th/9811056`.

[13] G. W. Gibbons, H. Lü, D. N. Page, and C. N. Pope, "The general kerr-de sitter metrics in all dimensions," *J. Geom. Phys.* **53** (2005) 49–73, `hep-th/0404008`.

[14] G. W. Gibbons, M. J. Perry, and C. N. Pope, "The first law of thermodynamics for kerr - anti-de sitter black holes," `hep-th/0408217`.

[15] M. M. Caldarelli and D. Klemm, "Supersymmetry of anti-de sitter black holes," *Nucl. Phys.* **B545** (1999) 434–460, `hep-th/9808097`.

[16] M. M. Caldarelli, "On supersymmetric solutions of d = 4 gauged supergravity," `hep-th/0411153`.

[17] P. G. O. Freund and M. A. Rubin, "Dynamics of dimensional reduction," *Phys. Lett.* **B97** (1980) 233–235.

[18] J. M. Figueroa-O'Farrill, F. Leitner, and J. Simón, "Supersymmetric freund–rubin backgrounds," `to appear`.

[19] J. P. Gauntlett, D. Martelli, J. Sparks, and D. Waldram, "Sasaki-einstein metrics on s(2) x s(3)," `hep-th/0403002`.

[20] J. P. Gauntlett, D. Martelli, J. F. Sparks, and D. Waldram, "A new infinite class of sasaki-einstein manifolds," `hep-th/0403038`.

[21] Y. Hashimoto, M. Sakaguchi, and Y. Yasui, "Sasaki-einstein twist of kerr-ads black holes," *Phys. Lett.* **B600** (2004) 270–274, `hep-th/0407114`.

[22] P. van Nieuwenhuizen and N. P. Warner, "Integrability conditions for killing spinors," *Commun. Math. Phys.* **93** (1984) 277.

[23] We thank Chris Pope for the communication on this point.